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On Rational Interpolation to Meromorphic Functions in Several Variables

Zebenzuí García[1]

Departamento de Análisis Matemático, Universidad de la Laguna,
38271 La Laguna, Tenerife, Spain
E-mail: zegarcia@ull.es

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In this paper, a new approach to construct rational interpolants to functions of several variables is considered. These new families of interpolants, which in fact are particular cases of the so-called Padé-type approximants (that is, rational interpolants with prescribed denominators), extend the classical Padé approximants (for the univariate case) and provide rather general extensions of the well-known Montessus de Ballore theorem for several variables. The accuracy of these approximants and the sharpness of our convergence results are analyzed by means of several examples. © 2000 Academic Press

1. INTRODUCTION

Because of the growing interest in rational approximation for its many applications (see, e.g., [4, Vol. 14]), during the past 25 years several works have been devoted to the extension of the Padé Approximation to functions of several variables. The classical paper by Chisholm [8], where equidiagonal Padé approximants (PA) to bivariate functions are constructed, could be considered as the starting point of these works. Then, in papers by the so-called Canterbury Group (see, for instance, [9, 23], Karlsson and Wallin (see [24, 39]), Lutterodt [28], and Cuyt [11, 13], among others, different ways to extend this topic to multivariate functions have been analyzed. In [13], most of these definitions are included as particular cases of a general multivariate Padé framework, by means of a unified treatment.

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According to [13], general order multivariate Padé approximants can be defined as follows. Indeed, given a function f holomorphic on a neighborhood of the origin in $\mathbb{C}^d$ ($d \in \mathbb{N}$), which admits a power series expansion of the form

$$f(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha z^\alpha \tag{1.1}$$

and two finite subsets $N, M \subset \mathbb{N}^d$ (that in this paper we will call lattices) of respective cardinality $\# N = n+1$ and $\# M = m+1$, there exist polynomials $p(z)$ and $q(z)$ whose exponents belong to the lattices N and M, respectively, and satisfy

$$(fq - p)(z) = \sum_{\alpha \in \mathbb{N}^d \setminus I} e_\alpha z^\alpha \tag{1.2}$$

provided that, in analogy with the univariate case, the equation lattice I satisfies:

(1.3a) $N \subset I$

(1.3b) $\#(I \setminus N) = m = \# M - 1$

(1.3c) I satisfies the so-called inclusion property (or rectangle rule) (see, e.g., [13]).

Recall that (1.3c) guarantees that (1.2) is actually a Hermite interpolation problem (without gaps). In the following, we make use of the standard multi-index notation: that is, for $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, $z = (z_1, ..., z_d) \in \mathbb{C}^d$, $v = (v_1, ..., v_d) \in (\mathbb{R} \setminus \{0\})^d$, and $\lambda \geqslant 0$ we denote $\alpha! = \alpha_1! \cdots \alpha_d!$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\Pi_j(z) = z_j$, $\Pi_j(\alpha) = \alpha_j$, $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, and $\lambda^v = (\lambda^{v_1}, ..., \lambda^{v_d})$. Furthermore, for any $z, z' \in \mathbb{C}^d$, we will write $\langle z', z \rangle = \sum_{i=1}^d z'_i \overline{z_i}$ and $zz' = (z_1 z'_1, ..., z_d z'_d)$.

In the same way, for two given sets $A, B \subset \mathbb{N}^d$, the sum of these sets (related to the set of exponents corresponding to the product of two polynomials) is defined by:

$$A + B = \{(a+b) : a \in A, b \in B\}.$$

Analogously, the "difference" set is given by:

$$A - B = \{(a-b) : a \in A, b \in B\} \cap \mathbb{N}^d.$$

Most likely, the first natural test to check the goodness of a class of multivariate PA is trying to prove an extension of the classical de Montessus de Ballore theorem, related to the convergence of rows of PA to meromorphic functions (see [20]). Different ways to obtain such extension are due to Chisholm and Graves-Morris [10], Graves-Morris [21], Karlsson and Wallin [24], Lutterodt [29, 30], and Cuyt ([12] for homogeneous multivariate PA and [14, 15] for the general case). However, none of these approaches seem to be rather general. Moreover, some counterexamples given in [24] and [39] point out that the obstacles for such general extension are not merely the methods of proof. Perhaps this difficulty in the extension is another sign of the well-known amount of troubles when passing from one to several complex variables ("the life is harder in $\mathbb{C}^n$," as is pointed out by a collection of several classic problems in the introduction of the book by Krantz [27]).

As indicated by Karlsson and Wallin in [24], the key point in the simplicity of the proof of the Montessus theorem in the univariate case (see, for instance, the simple proof shown in [38]) is the following. If we consider a function f holomorphic in a neighborhood of the origin and meromorphic in a disk D centered at this point, with precisely v poles counting multiplicities, and denote by Q the monic polynomial of degree v with zeros at these poles, then if we take p_n/q_v the $[n/v]$ Padé approximant to f, for which

$$(fq_v - p_n)(z) = \sum_{j>n+v} d_j z^j,$$

it is clear that

$$(fQq_v - Qp_n)(z) = \sum_{j>n+v} \tilde{e}_j z^j. \tag{1.4}$$

Moreover, the terms of the error series in the right-hand in right-hand term in (1.4) are not influenced by the coefficients of Qp_n, since this is a polynomial of degree less than or equal to $(n+v)$. However, as Karlsson and Wallin pointed out, the situation is quite different in the multivariate case, where in general we cannot ensure that $N+M \subset I$. It is not too clear how this problem is solved in [14, 15, 21, 29, 30], although it seems to be recognized in some recent papers see [16]), where the requirement $N+M \subset I$ is imposed in order to guarantee the convergence theorem. However, this is a very unnatural condition, and the reader can easily check the problems for constructing sequences of multivariate PA satisfying this condition on the equations lattice I.

Therefore, when passing to several variables, we have that the right-hand term in (1.4) is influenced by terms of Qp_n (and the number of these "naughty" guests probably increases with n). The natural question that arises at this point is how we can control the influence of these terms. A natural way to do it is to select the denominators of our PA in order to minimize the size of these terms in a certain sense (instead of leaving this denominator free as in the classical Padé approach). This requirement on the denominators is the basis of the so-called Padé-type approximants (PTA), terminology introduced by Brezinski (see [6, 7]) and used by many authors (for a classical and extensive study of rational interpolants with prescribed poles we refer to [40]). The idea is to make use of certain information about the knowledge of the singularities of the function (or, in our case, certain desirable minimizing property) to prescribe the denominators in order to guarantee good convergence results. In the multivariate case, some works have treated the PTA (see [1, 2, 5, 25, 36, 37]) and their convergence properties [1, 19, 31, 35]. For instance, we can take from [1] the definition of multivariate general order Padé-type approximants. Indeed, given a polynomial q with exponents belonging to the finite subset (denominator lattice) $M \subset \mathbb{N}^d$, and given a finite subset (numerator lattice) $N \subset \mathbb{N}^d$, there exists a unique polynomial p, with exponent set N, such that

$$(fq - p)(z) = \sum_{\alpha \in \mathbb{N}^d \backslash N} e_\alpha z^\alpha. \tag{1.5}$$

Then, the rational function (p/q) will be said to be an (N/M) multivariate PTA to f.

As we have said above, in the following we will construct sequences of PTA, selecting the denominators in order to minimize the size of the undesirable terms above to provide a general extension of the Montessus theorem. However, and in spite of the fact that these new optimal rational approximants will be properly PTA (and not PA), we shall show, by means of the extension property, that in the univariate case they agree with the classical PA. Therefore, in this sense (but perhaps only in this), these optimal PTA can also claim the name of multivariate PA.

The outline of the paper is the following. In Section 2, the precise definition of these optimal Padé-type approximants (OPTA) is given and their main algebraic properties (extension, consistency) stated. Furthermore, general extensions of the Montessus theorem (main results) are also deduced. In Section 3, the proofs of the different results are developed. Finally, in Section 4, several numerical examples are displayed, showing the power of this new class of rational approximants. On the other hand, a couple of counterexamples are analyzed in order to prove the sharpness of our main results.

2. DEFINITIONS AND RESULTS

2.1. *Algebraic Aspects*

As mentioned in the previous section, we are now concerned with the construction of our new class of rational approximants (actually, a class of multivariate Padé-type approximants). Let us start with the following

DEFINITION 2.1. Let $T: \mathbb{C}^m \to \mathbb{C}^n$ be a linear mapping with $m \in \mathbb{N} \backslash \{0\}$ and $n \in \mathbb{N}$ and let δ be a real number such that $\delta \geqslant 1$. Then, $x = (x_1, ..., x_m) \in \mathbb{C}^m$ is said to be a strong pseudominimum of T for $[m, n, \delta]$ with respect to a certain norm $\|\cdot\|$ in $\mathbb{C}^n$ if $x_1 = 1$ and

$$\|Tx\| \leqslant \delta \min_{y_1 = 1} \|Ty\|.$$

In a similar way, we say that $x \in \mathbb{C}^m$ is a weak pseudominimum of T for $[m, n, \delta]$ with respect to the norms $\|\cdot\|$ in $\mathbb{C}^n$ and $\|\cdot\|_*$ in $\mathbb{C}^m$ if $\|x\|_* = 1$ and

$$\|Tx\| \leqslant \delta \min_{\|y\|_* = 1} \|Ty\|.$$

Remark 2.1. Under the conditions of Definition 2.1, there exists, at least, a strong pseudominimum (weak pseudominimum respectively) of T for $[m, n, \delta]$ with respect to a certain norm $\|\cdot\|$ in $\mathbb{C}^n$ (resp. with respect to the norms $\|\cdot\|$ in $\mathbb{C}^n$ and $\|\cdot\|_*$ in $\mathbb{C}^m$). It is clear since there exists at least a strong pseudominimum (resp. weak pseudominimum) of T for $[m, n, 1]$ with respect to the same norms.

Remark 2.2. We point out the difference between strong and weak pseudominimums of a linear mapping in order to extend the definition of classical Padé approximants (in the strong or weak sense, see [4, Vol. 13]).

Now, our purpose is to show that the definition of pseudominimum does not depend, in a certain sense, on the norm used. For it, we shall deal with sequences of linear mappings. Indeed, consider the sequences $(m_k)_{k \in \mathbb{N}} \subset \mathbb{N} \backslash \{0\}$, $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$, $(\delta_k)_{k \in \mathbb{N}} \subset [1, \infty)$, $(s_k)_{k \in \mathbb{N}} \subset [1, \infty]$, and $(T_k)_{k \in \mathbb{N}}$ a sequence of linear mappings with $T_k: \mathbb{C}^{m_k} \to \mathbb{C}^{n_k}$. In the following, we denote for $x = (x_1, ..., x_m) \in \mathbb{C}^m$ and $1 \leqslant p < \infty$, $\|x\|_p = \{\sum_{i=1}^m |x_i|^p\}^{1/p}$ and $\|x\|_\infty = \max_{1 \leqslant i \leqslant m} |x_i|$. Now, we can state

PROPOSITION 2.1. *Consider the sequence $(x_k)_{k \in \mathbb{N}}$, where, for each k, x_k is a strong pseudominimum of T_k for $[m_k, n_k, \delta_k]$ with respect to the ℓ_p-norm $\|\cdot\|_{s_k}$ in $\mathbb{C}^{n_k}$. We denote $n'_k = \max\{1, n_k\}$ and $\delta'_k = n'_k \delta_k \geqslant 1$. If the sequence $(\sigma(k))_{k \in \mathbb{N}}$ in $(0, \infty]$ is such that $\lim_{k \to \infty} (\delta_k)^{1/\sigma(k)} = 1 = \lim_{k \to \infty} (n'_k)^{1/\sigma(k)}$, then, for each k, x_k is a strong pseudominimum of T_k for $[m_k, n_k, \delta'_k]$ with respect to the norm $\|\cdot\|_1$ (in $\mathbb{C}^{n_k}$), and $\lim_{k \to \infty} (\delta'_k)^{1/\sigma(k)} = 1$.*

Remark 2.3. Moreover, if under the same conditions, we have that $\lim_{k\to\infty}(m_k)^{1/\sigma(k)}=1$ and, for each $k\in\mathbb{N}$, x_k is a weak pseudominimum of T_k for $[m_k, n_k, \delta_k]$ with respect to the ℓ_p-norms $\|\cdot\|_{s_k}$ in $\mathbb{C}^{n_k}$ and $\|\cdot\|_{t_k}$ in $\mathbb{C}^{m_k}$ (also, $t_k\in[1,\infty]$, $k\in\mathbb{N}$), then if we denote by $n'_k=\max\{1, n_k\}$ and $\delta'_k=n'_k m_k\delta_k\geqslant 1$, we have that $x_k/\|x_k\|_\infty$ is a weak pseudominimum of T_k for $[m_k, n_k, \delta'_k]$ with respect to $\|\cdot\|_1$ (in $\mathbb{C}^{n_k}$) and $\|\cdot\|_\infty$ (in $\mathbb{C}^{m_k}$), and $\lim_{k\to\infty}(\delta'_k)^{1/\sigma(k)}=1$.

After these preliminary considerations, we can now give the definition of our class of multivariate Padé-type approximants, which will be called *optimal Padé-type approximants* (OPTA). For this, let $d\in\mathbb{N}\setminus\{0\}$ and consider a (possibly formal) power series $f(x)=\sum_{\alpha\in\mathbb{N}^d} f_\alpha x^\alpha$.

DEFINITION 2.2. If N, M are two finites subsets in $\mathbb{N}^d$ with $0\in M$, R is a polyradius $R>0$ (here and in the remainder of the paper it means that $R=(R_1, ..., R_d)$ with $R_i>0$, for $i=1, ..., d$) and $\delta\geqslant 1$, we say that the rational function r is a strong OPTA of f for $[N, M, R, \delta]$ if the following holds,

(a) $r=p/q$, with $p\in\pi_N$, $q\in\pi_M$ (if L is a finite subset of $\mathbb{N}^d$ and t is a polynomial, now and in the remainder of the paper, the notation $t\in\pi_L$ means that L is the exponent set of t).

(b) Considering the set $E=E(N, M)=((N+M)-M)\setminus N$ and setting $q(x)=\sum_{\beta\in M} q_\beta x^\beta$ and the linear function $T\colon\mathbb{C}^{\#M}\to\mathbb{C}^{\#E}$, such that for $M\neq\{0\}$ maps the vector $u=(u_\beta)_{\beta\in M}$ onto the vector $v=(\sum_{\beta\in M} u_\beta f_{\alpha-\beta} R^\alpha)_{\alpha\in E}$ then the vector $(q_\beta)_{\beta\in M}$ is a strong pseudominimum of T for $[\#M, \#E, \delta]$ with respect to the norm $\|\cdot\|_1$ in $\mathbb{C}^{\#E}$, where now we take u_0 as the first component of $(u_\beta)_{\beta\in M}\in\mathbb{C}^{\#M}$.

(c) p is the Taylor polynomial of the function fq with N as its exponent set; that is, $(fq-p)(x)=\sum_{\alpha\in\mathbb{N}^d\setminus N} e_\alpha x^\alpha$.

Remark 2.4. Under the same conditions, we say that r is a weak OPTA of f for $[N, M, R, \delta]$, when the requirements above are satisfied, but now, in (b) the vector $(q_\beta)_{\beta\in M}$ is taken as a weak pseudominimum of T for $[\#M, \#E, \delta]$ with respect to the norm $\|\cdot\|_1$ in $\mathbb{C}^{\#E}$ and the norm $\|\cdot\|_\infty$ in $\mathbb{C}^{\#M}$.

Remark 2.5. From Remark 2.1 it follows the existence of a strong (resp. weak) OPTA of f for $[N, M, R, \delta]$.

Remark 2.6. If $M=\{0\}$, then $E=\varnothing$ and in this case the unique linear mapping $T\colon\mathbb{C}\to\mathbb{C}^0=\{0\}$ is the null function, and hence the unique OPTA of f is its Taylor polynomial with exponent set N.

Remark 2.7. If N and M satisfy the rectangle rule, then $E = (N+M)\setminus N$.

Remark 2.8. It can be seen that the definition above is clearly motivated by the purpose of minimizing the influence of the "naughty" terms corresponding to the indexes in $(N+M)\setminus I$, as we pointed out in the introduction.

From the definition above, it seems that for a large δ, almost every vector could be a pseudominimum. Hence, many rational functions could be OPTA. However, and taking into account that the main aim of this paper is the extension of the de Montessus de Ballore theorem, in order to establish the uniform (and geometrical) convergence of certain sequences of OPTA to meromorphic functions, this broadness in the definition can be avoided with the following definition of geometrical sequences of OPTA.

DEFINITION 2.3. Let f and R be as above, $(N_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$ be two sequences of finite subsets in $\mathbb{N}^d$ with $0 \in M_k$ for each k, and $\sigma = (\sigma(k))_{k\in\mathbb{N}}$ and $\delta = (\delta_k)_{k\in\mathbb{N}}$ two sequences of real numbers in $(0, \infty]$ and $[1, \infty)$, respectively, such that $\lim_{k\to\infty} \sigma(k) = \infty$ and $\lim_{k\to\infty} (\delta_k)^{1/\sigma(k)} = 1$. A sequence of rational functions $(r_k)_{k\in\mathbb{N}}$ is said to be a σ-geometrically strong (weak) OPTA of f for $[(N_k)_{k\in\mathbb{N}}, (M_k)_{k\in\mathbb{N}}, R, \delta, \sigma]$ if for each $k \in \mathbb{N}$, r_k is a strong (weak) OPTA of f for $[N_k, M_k, R, \delta_k]$.

On the other hand, in spite of the fact that from the definition of OPTA the computational viability of these approximants does not seem to be clear, now we will show how they can be easily computed.

Remark 2.9. As in the definition of sequences of strong (weak) pseudominimums, it is easy to prove that this definition does not depend on the norms; that is, we can replace the norm $\|\cdot\|_1$ in $\mathbb{C}^{\# E_k}$ (and the norm $\|\cdot\|_\infty$ in $\mathbb{C}^{\# M_k}$, for the weak case) for any sequence of ℓ_p-norms $\|\cdot\|_{s_k}$ in $\mathbb{C}^{\# E_k}$, where for each k, $s_k \in [1, \infty]$ (for any sequence of ℓ_p-norms $\|\cdot\|_{t_k}$ in $\mathbb{C}^{\# M_k}$, where for each k, $t_k \in [1, \infty]$), provided that $\lim_{k\to\infty} (\max\{1, \# E_k\})^{1/\sigma(k)} = 1$ (if $\lim_{k\to\infty} (\max\{1, \# E_k\})^{1/\sigma(k)} = 1$ then $\lim_{k\to\infty} (\# M_k)^{1/\sigma(k)} = 1$).

Indeed, in practice (see the numerical examples displayed in Section 4), these OPTA can be computed by a straight forward procedure, since their denominators arise as least squares solutions of inconsistent systems of linear equations. The reason for dealing with the general concept of pseudominimum is to provide an extension of the classical de Montessus de Ballore theorem for several variables as general as possible.

Now, we are going to state two basic algebraic properties of these OPTA (a deeper algebraic study is left for a further paper). The first of them extends the so-called consistency property of the classical PA, that is, the fact that every OPTA for a rational function f, corresponding to sufficiently large lattices N and M, is identical to f. The second one, which we can call the extension property, asserts that in the univariate case our OPTA agrees with the classical PA. Then, in this sense, the OPTA can claim to be a natural generalization of the univariate PA.

PROPOSITION 2.2 (Consistency). *Let p and q be two multivariate polynomials with respective exponent sets given by N' and M', with $0 \in M'$ and $q(0) \neq 0$ (resp. $q \neq 0$) such that $f(x) = p(x)/q(x)$ is holomorphic in a neighborhood of the origin. If $N, M \subset \mathbb{N}^d$ are two finite subsets so that $N' \subseteq N$ and $M' \subseteq M$, then, for each $R > 0$ and $\delta \geqslant 1$, f is the unique strong (weak) OPTA of f for $[N, M, R, \delta]$.*

PROPOSITION 2.3 (Extension). *Let us consider a power series $f(x) = \sum_{k=0}^{\infty} f_k x^k$. Then, the $[n/m]$ classical strong (weak) PA to f is the unique strong (weak) OPTA for $[N, M, R, \delta]$, with $N = \{0, 1, ..., n\}$, $M = \{0, 1, ..., m\}$, and any $R > 0$ and $\delta \geqslant 1$.*

Remark 2.10. Observe that our strong OPTA extends the classical univariate PA (in the sense of Baker's definition, see, e.g., [4, Vol. 13]), when it exists. However, in the case where the PA exists only in the weak sense, our strong OPTA provides a new version of strong PA which does not agree with the weak version. These implications will be investigated in a forthcoming paper.

2.2. *Convergence Theorems*

Now, we state our main results, related to the extension of the de Montessus de Ballore theorem for the OPTA. First, let us specify some notations.

Indeed, let $(N_k)_{k \in \mathbb{N}}$, $(M_k)_{k \in \mathbb{N}}$, and $(E_k)_{k \in \mathbb{N}} \subset \mathbb{N}^d$ be as above, where $E_k = E(N_k, M_k)$. For any vector $v \in (\mathbb{R}^+)^d$ denote by $\Lambda_v(k) = \min\{\langle v, \alpha\rangle : \alpha \in \mathbb{N}^d \backslash N_k\}$ and $\sigma_v(k) = \min\{\langle v, \alpha\rangle : \alpha \in E_k\}$ (if $E_k = \emptyset$ then $\sigma_v(k) = \Lambda_v(k)$), where for simplicity we write $\Lambda_1(k) = \Lambda_{(1, ..., 1)}(k)$ and $\sigma_1(k) = \sigma_{(1, ..., 1)}(k)$.

Now, it should be indicated that our convergence results will be established for compact subsets of the domain of convergence of the Taylor series of fQ, where f is a meromorphic function and Q is a polynomial making fQ holomorphic. It is well known that the convergence domain for a Taylor series in $\mathbb{C}^d$ is a logarithmically convex complete Reinhardt domain (see, e.g., [34] for the definition). In the rest of the paper, for a complete Reinhardt domain $\mathfrak{D}$ in $\mathbb{C}^d$, a vector $v \in (\mathbb{R}^+)^d$, and a polyradius $R > 0$, we shall denote by $\rho_v(R, \mathfrak{D}) = \inf\{\lambda > 0 : P(0, R\lambda^{-v}) \subset \mathfrak{D}\}$ where

$P(z, r)$ denotes the polydisc centered in $z \in \mathbb{C}^d$ and with polyradius $r > 0$ (observe that if $\overline{P(0, R)} \subset \mathfrak{D}$, then $\rho_v(R, \mathfrak{D}) < 1$).

On the other hand, for $\gamma \in \mathbb{N}^d$, we make use of the notation $D^\gamma = \partial^{\gamma_1}/\partial x_1^{\gamma_1} \cdots \partial^{\gamma_d}/\partial x_d^{\gamma_d}$ to denote partial derivatives, and finally, for any polynomial Q in d variables, we denote $L_\varepsilon = \{x \in \mathbb{C}^d : |Q(x)| < \varepsilon\}$.

THEOREM 2.4. *Let f be a holomorphic function in a neighborhood of the origin in $\mathbb{C}^d$ and $Q(x) = \sum_{\beta \in M} Q_\beta x^\beta$ a polynomial, with exponent set $M \subset \mathbb{N}^d$, $0 \in M$ and $Q(0) > 0$ ($Q_{\beta_0} > 0$, for some $\beta_0 \in M$, respectively). Let $\mathfrak{D}$ be the domain where the Taylor expansion of Qf converges and take a polyradius $R > 0$ such that $\overline{P(0, R)} \subset \mathfrak{D}$. Let $(N_k)_{k=1}^\infty$ be a sequence of finite sets in $\mathbb{N}^d$ such that $\lim_{k\to\infty} \Lambda_1(k) = \infty$ and $(r_k)_{k=1}^\infty$ be a σ_1-geometrically strong (resp. weak) OPTA of f for $[(N_k)_{k=1}^\infty, (M)_{k=1}^\infty, \delta, R, \sigma_1]$, where δ is taken such that $\lim_{k\to\infty} (\delta_k)^{1/\sigma_1(k)} = 1$. Finally, suppose that there exist h points $z_1, ..., z_h \in \overline{P(0, R)}$ and h finite sets $\mathrm{I}_1, ..., \mathrm{I}_h \subset \mathbb{N}^d$ satisfying the rectangle rule and such that*

$$\left.\begin{array}{l} \times\, D^\gamma Q(z_i) = 0, \qquad \gamma \in \mathrm{I}_i, \qquad 1 \leqslant i \leqslant h. \\ \times\, (fQ)(z_i) \neq 0 \qquad 1 \leqslant i \leqslant h \\ \times\, \sum_{i=1}^h \# \mathrm{I}_i = \# M - 1 \\ \times\, \det\left[\left(\frac{\beta!}{(\beta-\gamma)!}\, z_i^{\beta-\gamma}\right)_{\gamma \in \mathrm{I}_i,\ 1 \leqslant i \leqslant h;\ \beta \in M^* = M \setminus \{0\}\ (= M \setminus \{\beta_0\}\ \text{resp.})}\right] \neq 0. \end{array}\right\} \tag{2.1}$$

Then:

(i) *If for each pair (i, j), $1 \leqslant i \leqslant h$ and $1 \leqslant j \leqslant d$, such that $|\Pi_j(z_i)| = R_j$, $\Pi_j(\mathrm{I}_i) = \{0\}$ holds, then for each $v \in (\mathbb{R}^+)^d$ and $\mu \in [0, 1]$ we have for any $\varepsilon > 0$*

$$\overline{\lim_{k\to\infty}} \left(\|f - r_k\|_{\infty, \overline{P(0, r)} \setminus L_\varepsilon}\right)^{1/\Lambda_v(k)} \leqslant \rho_v(r, \mathfrak{D}) < 1, \tag{2.2}$$

where $r = R\mu^v$.

(ii) *If the hypothesis* (i) *is not necessarily fulfilled, but we have that $\overline{\lim}_{k\to\infty} (\max\{\alpha_i : \alpha \in E_k, 1 \leqslant i \leqslant d\})^{1/\sigma_1(k)} = 1$, then for any $v \in (\mathbb{R}^+)^d$, $\gamma \in \mathbb{N}^d$ and $\mu \in [0, 1]$, one has for each $\varepsilon > 0$*

$$\overline{\lim_{k\to\infty}} \left(\|D^\gamma(f - r_k)\|_{\infty, \overline{P(0, r)} \setminus L_\varepsilon}\right)^{1/\Lambda_v(k)} \leqslant \rho_v(r, \mathfrak{D}) < 1, \tag{2.3}$$

where $r = R\mu^v$.

(iii) *Furthermore, provided that the assumptions* (i) *or* (ii) *hold, if we denote* $t_i = \min\{t \in [0, 1] : z_i \in P(0, Rt^v)\}$ *and* $\mathscr{L}_v = R(\max_{1 \leqslant i \leqslant h} t_i)^v$, *one has*

$$\overline{\lim_{k \to \infty}} \, (\|q_k - Q\|)^{1/A_v(k)} \leqslant \rho_v(\mathscr{L}_v, \mathfrak{D}) < 1, \tag{2.4}$$

where Q *and* q_k *are suitably normalized.*

THEOREM 2.5. *Under the same conditions as in Theorem* 2.4, *for* $u \in (\mathbb{R}^+)^d$ *denote* $S_u(k) = \max\{\langle u, \alpha\rangle : \alpha \in E_k\}$ (*if* $E_k = \varnothing$ *we denote* $S_u(k) = \sigma_u(k)$) *and suppose that* $\lim_{k \to \infty} S_u(k)/\sigma_u(k) = 1$. *Now, assume that there exist* $\lambda_0 \in \mathbb{R}^+$ *and h points* $z_1, ..., z_h \in \overline{P(0, R\lambda_0^u)} \subset \mathfrak{D}$ *and h finite sets* $\mathrm{I}_1, ..., \mathrm{I}_h$ *in* $\mathbb{N}^d$ *satisfying the rectangle rule and such that* (2.1) *holds.*

Then, for each $\lambda \in \mathbb{R}^+$ *with* $\overline{P(0, R\lambda^u)} \subset \mathfrak{D}$, *and for each* $v \in (\mathbb{R}^+)^d$ *and* $\mu \in [0, 1]$, *one has for every* $\varepsilon > 0$

$$\overline{\lim_{k \to \infty}} \, (\|f - r_k\|_{\infty, \overline{P(0, r)} \setminus L_\varepsilon})^{1/A_v(k)} \leqslant \rho_v(r, \mathfrak{D}) < 1, \tag{2.5}$$

where $r = R\lambda^u \mu^v$.

Moreover, in both strong and weak cases, for each $\lambda \in \mathbb{R}^+$ *verifying that* $\{z_1, ..., z_h\} \subset \overline{P(0, R\lambda^u)} \subset \mathfrak{D}$ *and* $v \in (\mathbb{R}^+)^d$, *one has*

$$\overline{\lim_{k \to \infty}} \, (\|Q - q_k\|)^{1/A_v(k)} \leqslant \rho_v(\mathscr{L}_{\lambda, v}, \mathfrak{D}) < 1, \tag{2.6}$$

where Q *and* q_k *are suitably normalized, with* $t_i = \min\{t \in [0, 1] : z_i \in \overline{P(0, R\lambda^u t^v)}\}$ $(1 \leqslant i \leqslant h)$ *and* $\mathscr{L}_{\lambda, v} = R\lambda^u(\max_{1 \leqslant i \leqslant h} t_i)^v$.

Remark 2.11. Although the difference between the strong and the weak PA is not essential for the univariate Montessus theorem, the situation is not the same for several variables. Indeed, in the univariate case, if a function f (holomorphic in the origin) is such that Qf, with Q a polynomial, is holomorphic in a domain containing the origin, we can suppose that $Q(0) \neq 0$. This assumption does not hold, in general, for the multivariate case, as we can see by taking the function $f(x, y) = 1/(x + 2 + \sqrt{y + 4})$, $Q(x, y) = (x + 2)^2 - (y + 4)$ and the domain $\mathbb{C} \times \{\mathrm{Re}(y) > -4\}$. From this fact, we must distinguish between the strong and weak OPTA both in Theorems 2.4 and 2.5.

Remark 2.12. From the statements in Theorems 2.4 and 2.5 it is clear that we have extended the de Montessus de Ballore theorem to our OPTA, in the sense that we establish the uniform (even geometrical) convergence of sequences of such rational interpolants in polydiscs compactly contained in the logarithmically convex complete Reinhardt domain of the Taylor series of Qf. In the same way, we prove the geometrical convergence of the denominators of r_k.

In Theorem 2.4, we have total freedom to select the sequence $(N_k)_{k=1}^{\infty}$, but we need to choose a suitable polyradius R such that $\overline{P(0, R)} \subset \mathfrak{D}$ and (2.1) holds, in order to ensure uniform (even geometrical) convergence in compact subsets of $\overline{P(0, R)} \backslash Q^{-1}(\{0\})$. Theorem 2.5 shows that if the natural condition $\lim_{k\to\infty} S_u(k)/\sigma_u(k) = 1$ is satisfied, we can guarantee geometrical convergence in compact subsets of the larger set

$$\bigcup_{0<\lambda<(\rho_u(R, \mathfrak{D}))^{-1}} \overline{P(0, R\lambda^u)} \backslash Q^{-1}(\{0\}),$$

with a relative independence on the choice of R. In fact, in the particular case when $d = 1$ (univariate case) and $M = \{0, 1, ..., m\}$, with $m = \# M - 1$, in order to apply Theorem 2.4 we have a total freedom to select the sequence $(N_k)_{k=1}^{\infty}$, but we must take a radius R belonging to the interval (R_{m-1}, R_m), where for each $n \in \mathbb{N}$, R_n denotes the n-meromorphy radius of f. In this situation, the convergence is achieved in compact subsets of $\overline{P(0, R)} \backslash Q^{-1}(\{0\})$. On the contrary, if the natural condition $\lim_{k\to\infty} S_1(k)/\sigma_1(k) = 1$ is satisfied, then by applying Theorem 2.5, we can guarantee convergence in compact subsets of the larger set $P(0, R_m) \backslash Q^{-1}(\{0\})$, for any radius $R > 0$. In this sense, we consider Theorem 2.5 as the proper extension of the univariate de Montessus de Ballore theorem, even when dealing with the univariate case.

Conditions (2.1) guarantee the unicity of Q and give the geometrical convergence of the denominators of r_k. Such conditions were previously used by Cuyt in [14, 15].

3. PROOFS

3.1. *Algebraic Aspects*

Proof (Proposition 2.1). (a) Since for each $k \in \mathbb{N}$, x_k is a (strong) pseudominimum of T_k for $[m_k, n_k, \delta_k]$ with respect to the ℓ_p-norm $\|\cdot\|_{s_k}$ in $\mathbb{C}^{n_k}$, we have

$$\|T_k x_k\|_1 \leqslant (n_k)^{(s_k-1)/s_k} \|T_k x_k\|_{s_k} \leqslant n_k \delta_k \min_{y_1=1} \|Ty\|_{s_k} \leqslant \delta'_k \min_{y_1=1} \|Ty\|_1.$$

Moreover, from the fact that $\lim_{k\to\infty}(\delta_k)^{1/\sigma(k)} = 1 = \lim_{k\to\infty}(n'_k)^{1/\sigma(k)}$, it is clear that $\lim_{k\to\infty}(\delta'_k)^{1/\sigma(k)} = 1$.

(b) Similarly, since for each $k \in \mathbb{N}$, x_k is a (weak) pseudominimum of T_k for $[m_k, n_k, \delta_k]$ with respect to the ℓ_p-norms $\|\cdot\|_{s_k}$ in $\mathbb{C}^{n_k}$ and $\|\cdot\|_{t_k}$ in $\mathbb{C}^{m_k}$, we obtain

$$\left\| T_k \frac{x_k}{\|x_k\|_\infty} \right\|_1 \leqslant \frac{1}{\|x_k\|_\infty} (n_k)^{(s_k-1)/s_k} \|T_k x_k\|_{s_k}$$
$$\leqslant \frac{\|x_k\|_{t_k}}{\|x_k\|_\infty} n_k \delta_k \min_{\|y\|_{t_k}=1} \|Ty\|_{s_k} \leqslant \delta'_k \min_{y_1=1} \|Ty\|_1.$$

Finally, since $\lim_{k\to\infty}(\delta_k)^{1/\sigma(k)} = 1 = \lim_{k\to\infty}(n'_k)^{1/\sigma(k)} = \lim_{k\to\infty}(m_k)^{1/\sigma(k)}$ it follows $\lim_{k\to\infty}(\delta'_k)^{1/\sigma(k)} = 1$. ∎

For the proof of the consistency and extension properties, we shall make use of the following

LEMMA 3.1. *Let $d \in \mathbb{N}\backslash\{0\}$ and consider a (possibly formal) power series $f(x) = \sum_{\alpha\in\mathbb{N}^d} f_\alpha x^\alpha$. Let M and N be two finite sets in $\mathbb{N}^d$, with $0 \in M$, and $q \in \pi_M$ a polynomial such that*

$$\begin{cases} \text{(i)} \quad q(0) \neq 0 & (\textit{resp. } q \neq 0) \\ \text{(ii)} \quad \left(\sum_{\beta\in M} q_\beta f_{\alpha-\beta} \right)_{\alpha\in E} = 0, & \textit{where} \quad E = ((N+M)-M)\backslash N. \end{cases}$$

Then, if p is the Taylor polynomial of the function fq with N as its exponent set, the rational function $r = p/q$ is the unique strong (resp. weak) OPTA of f for $[N, M, R, \delta]$ for any polyradius $R > 0$ and any $\delta \geqslant 1$.

Proof. Let $R > 0$ and $\delta \geqslant 1$ and denote by $\tilde{q}$ and $\tilde{p}$ the respective normalized polynomials corresponding to q and p, in the sense that $\tilde{q} \in \pi_M$, $\tilde{q}(0) = 1$ (resp. $\max_{\beta\in M} |\tilde{q}_\beta| = 1$) and $\tilde{p}$ is the Taylor polynomial of the function $f\tilde{q}$ with N as its set of exponents. So, we have that

$$\|T(\tilde{q}_\beta)_{\beta\in M}\|_1 = \left\| \left(\sum_{\beta\in M} \tilde{q}_\beta f_{\alpha-\beta} R^\alpha \right)_{\alpha\in E} \right\|_1 = 0$$
$$= \delta \min_{y_0=1} \|T(y_\beta)_{\beta\in M}\|_1 \; \left(= \delta \min_{\max_{\beta\in M}|y_\beta|=1} \|T(y_\beta)_{\beta\in M}\|_1, \text{ resp.}\right).$$

Therefore, $r = p/q = \tilde{p}/\tilde{q}$ is a strong (resp. weak) OPTA of f for $[N, M, R, \delta]$, for any $R > 0$ and $\delta \geqslant 1$. Now, suppose that r' is another strong (resp. weak) OPTA of f for $[N, M, R, \delta]$ with a polyradius $R > 0$ and $\delta \geqslant 1$. If we write $r' = p/q'$, where p' and q' are given by Definition 2.2 (resp. Remark 2.3), then $(\sum_{\beta\in M} q'_\beta f_{\alpha-\beta})_{\alpha\in E} = 0$. Hence,

$$(f\tilde{q} - \tilde{p})(x) = \sum_{\alpha\in\mathbb{N}^d\backslash((N+M)-M)} \tilde{c}_\alpha x^\alpha \qquad \text{and}$$
$$(fq' - p')(x) = \sum_{\alpha\in\mathbb{N}^d\backslash((N+M)-M)} c'_\alpha x^\alpha.$$

It is clear that

$$(\tilde{q}p' - q'\tilde{p})(x) = (q'(f\tilde{q} - \tilde{p}) - \tilde{q}(fq' - p'))(x) = \sum_{\alpha \in \mathbb{N}^d \setminus (N+M)} e_\alpha x^\alpha.$$

Since $(\tilde{q}p' - q'\tilde{p}) \in \pi_{N+M}$, one has that $\tilde{q}p' - q'\tilde{p} = 0$, and consequently $r = r'$. ∎

Proof (Proposition 2.2). We have that N', $M' \subset \mathbb{N}^d$ and $p \in \pi_{N'}$, $q \in \pi_{M'}$ with $q(0) \neq 0$ (resp. $q \neq 0$). Since $E \subset \mathbb{N}^d \setminus N \subset \mathbb{N}^d \setminus N'$ and $(fq)(x) = \sum_{\alpha \in N'} c_\alpha x^\alpha$, then $(\sum_{\beta \in M} q_\beta f_{\alpha - \beta})_{\alpha \in E} = 0$. Therefore, the proof easily follows by applying Lemma 3.1. ∎

Proof (Proposition 2.3). Consider the rational function $R(z) = A_{(n,m)}(z)/B_{(n,m)}(z)$, where the polynomials $A_{(n,m)}(z) \in \pi_n$ and $B_{(n,m)}(z) \in \pi_m$ satisfy:

$$\begin{cases} \text{(i)} \quad B_{(n,m)}(0) = 1, & \text{if there exists the } [n/m] \text{ strong PA} \\ \qquad\qquad (B_{n,m} \neq 0 \text{ for the weak case}) & \\ \text{(ii)} \quad (B_{(n,m)} f - A_{(n,m)})(z) = O(z^{n+m+1}) & \end{cases}$$

Setting $B_{(n,m)}(z) = \sum_{j=0}^m B_{(n,m),j} z^j$, then we have that $B_{(n,m)}(0) = 1$ if the $[n/m]$ PA exists in the strong sense (resp. $B_{(n,m)} \neq 0$). Moreover, $\sum_{j=0}^m B_{(n,m),j} f_{k-j} = 0$, for $k = n+1, ..., n+m$ and $A_{(n,m)}$ is the nth Taylor polynomial of f. Thus, a proof can be reached by a straightforward application of Lemma 3.1. ∎

3.2. *Convergence Theorems*

First, we need the following algebraic result.

LEMMA 3.2. *Let $d \in \mathbb{N} \setminus \{0\}$ and $\mathrm{I} \subset \mathbb{N}^d$ be a finite set verifying the inclusion property. Consider the mapping $L: \mathbb{C}^{\#\mathrm{I}} \to \mathbb{C}^{\#\mathrm{I}}$, $L((x_\theta)_{\theta \in \mathrm{I}}) = (\sum_{\theta \in \mathrm{I}} l(\beta, \theta)\, x_\theta)_{\beta \in \mathrm{I}}$ so that*

$$\begin{cases} L \text{ is } \mathbb{C}\text{-linear} \\ \text{For any } \theta \in \mathrm{I},\ l(\theta, \theta) \neq 0 \\ l(\beta, \theta) = 0, \qquad \text{for } \theta, \beta \in \mathrm{I} \text{ such that } \beta \ngeq \theta. \end{cases}$$

Then, the mapping L is univalent.

Proof. L possesses a nonsingular associated matrix, because (for a suitable basis) it is a triangular matrix with nonvanishing diagonal terms.

Proof (*Theorem* 2.4). Let $v \in (\mathbb{R}^+)^d$, $\mu \in [0, 1]$, $\lambda \in (\rho_v(R, \mathfrak{D}), 1)$, and set $r = R\mu^v$ and $\tilde{R} = R\lambda^{-v}$. For each k, consider $r_k = \tilde{p}_k / \tilde{q}_k$, where $\tilde{q}_k$ is the normalization of the polynomial q_k in order to satisfy that $1 = \max_{\beta \in M} |\tilde{q}_{\beta, k}|$ and $\tilde{q}_k(0) \geqslant 0$ (resp. $\tilde{q}_{\beta_0, k} \geqslant 0$).

Now, for a fixed $\gamma \in \mathbb{N}^d$ and $x \in \overline{P(0, r)}$ we have

$$D^\gamma(Qf\tilde{q}_k - Q\tilde{p}_k)(x)$$

$$= \underbrace{\left(\frac{1}{2\pi i}\right)^d \sum_{\alpha \in (N_k + M)^c} \frac{\alpha!}{(\alpha - \gamma)!} x^{\alpha - \gamma} \int_{b_0 P(0, \tilde{R})} \frac{(Qf\tilde{q}_k)(y)}{y^{\alpha + 1}} dy}_{A(x)}$$

$$+ \underbrace{D^\gamma \left(\left(\frac{1}{2\pi i}\right)^d \sum_{\alpha \in (N_k + M)} x^\alpha \int_{b_0 P(0, \tilde{R})} \frac{(Q(f\tilde{q}_k - \tilde{p}_k))(y)}{y^{\alpha + 1}} dy \right)}_{B(x)}.$$

Now, we have

$$|A(x)| \leqslant \|Qf\|_{\infty, P(0, \tilde{R})} \|\tilde{q}_k\|_{\infty, P(0, \tilde{R})} r^{-\gamma} \sum_{\alpha \in \mathbb{N}^d \setminus (N_k + M)} \frac{\alpha!}{(\alpha - \gamma)!} (\mu\lambda)^{\langle \alpha, v \rangle}$$

$$\leqslant \text{const } \|Qf\|_{\infty, P(0, \tilde{R})} \sum_{\alpha \in \mathbb{N}^d \setminus (N_k + M)} \frac{\alpha!}{(\alpha - \gamma)!} (\mu\lambda)^{\langle \alpha, v \rangle}.$$

Then

$$\overline{\lim_{k \to \infty}} \left(\sup\{|A(x)| : x \in \overline{P(0, r)}\}\right)^{1/A_v(k)} \leqslant \mu\lambda < 1, \qquad \gamma \in \mathbb{N}^d. \tag{3.1}$$

On the other hand, it can also be written

$$B(x) = D^\gamma \left(\sum_{\alpha \in E_k} c_\alpha x^\alpha \underbrace{\left(\sum_{\beta \in \{\beta \in M : \alpha + \beta \in (N_k + M)\}} Q^\beta x^\beta \right)}_{C(x)} \right)$$

$$= \sum_{\alpha \in E_k} c_\alpha \sum_{\theta \in \mathbb{N}^d, \theta \leqslant \gamma} \frac{\alpha!}{(\alpha - \theta)!} x^{\alpha - \theta} \frac{\gamma!}{(\gamma - \theta)!} D^{\alpha - \theta}[C(x)],$$

with $(f\tilde{q}_k - \tilde{p}_k)(x) = \sum_{\alpha \in N_k} c_\alpha x^\alpha$. Hence,

$$
\begin{aligned}
|B(x)| &\leqslant \text{const} \sum_{\alpha \in E_k} |c_\alpha|\, R^\alpha \mu^{\langle v, \alpha\rangle} \max_{\theta \leqslant \gamma} \left\{ \frac{\alpha!}{(\alpha-\theta)!} \right\} \\
&\leqslant \text{const}\, \mu^{\sigma_v(k)} \max_{\alpha \in E_k,\, \theta \leqslant \gamma} \left\{ \frac{\alpha!}{(\alpha-\theta)!} \right\} \sum_{\alpha \in E_k} \left| \sum_{\beta \in M} \tilde{q}_{\beta, k} f_{\alpha-\beta} \right| R^\alpha \\
&\leqslant \text{const}\, \dot{u}_k \mu^{\sigma_v(k)} \max_{\alpha \in E_k,\, \theta \leqslant \gamma} \left\{ \frac{\alpha!}{(\alpha-\theta)!} \right\} \sum_{\alpha \in E_k} \left| \sum_{\beta \in M} q_{\beta, k} f_{\alpha-\beta} \right| R^\alpha \\
&\leqslant \text{const}\, u_k \mu^{\sigma_v(k)} \delta(k) \max_{\alpha \in E_k,\, \theta \leqslant \gamma} \left\{ \frac{\alpha!}{(\alpha-\theta)!} \right\} \\
&\quad \times \sum_{\alpha \in E_k} \left| \sum_{\beta \in M} Q_\beta f_{\alpha-\beta} \right| R^\alpha \text{const}^* \\
&\leqslant \text{const} \max_{\alpha \in E_k,\, \theta \leqslant \gamma} \left\{ \frac{\alpha!}{(\alpha-\theta)!} \right\} \mu^{\sigma_v(k)} \delta(k)\, \|Qf\|_{\infty, P(0, \tilde{R})} \sum_{\alpha \in E_k} \lambda^{\langle v, \alpha \rangle}
\end{aligned}
$$

where $u_k = |\tilde{q}_k(0)|$ $(= \max_{\beta \in M} |\tilde{q}_{\beta, k}| = 1$, resp.), $\frac{1}{\text{const}^*} = Q(0)$ $(= \max_{\beta \in M} |Q_\beta|$, resp.)

Therefore,

$$
\begin{aligned}
&\overline{\lim_{k \to \infty}} \left(\sup\{|B(x)| : x \in \overline{P(0, r)}\}\right)^{1/\sigma_v(k)} \\
&\qquad \leqslant \mu\lambda \overline{\lim_{k \to \infty}} \left(\max_{\alpha \in E_k,\, \theta \leqslant \gamma} \left\{ \frac{\alpha!}{(\alpha-\theta)!} \right\} \right)^{1/\sigma_v(k)}.
\end{aligned} \tag{3.2}
$$

Now, from (3.2) and since $\Lambda_v(k) \leqslant \sigma_v(k)$ it follows that

$$
\overline{\lim_{k \to \infty}} \left(\sup\{|B(x)| : x \in \overline{P(0, r)}\}\right)^{1/\Lambda_v(k)} \leqslant \mu\lambda < 1, \qquad \text{if} \quad \gamma = 0. \tag{3.3}
$$

Thus, from (3.1) and (3.3), we obtain that

$$
\overline{\lim_{k \to \infty}} \left(\|Qf\tilde{q}_k - Q\tilde{p}_k\|_{\infty, P(0, r)} \right)^{1/\Lambda_v(k)} \leqslant \mu\lambda, \qquad \mu \in [0, 1], \quad \lambda \in (\rho_v(R, \mathfrak{D}), 1),
$$

and consequently

$$
\begin{aligned}
&\overline{\lim_{k \to \infty}} \left(\|Qf\tilde{k}_k - Q\tilde{p}_k\|_{\infty, P(0, r)} \right)^{1/\Lambda_v(k)} \\
&\qquad \leqslant \mu \rho_v(R, \mathfrak{D}) = \rho_v(r, \mathfrak{D}), \qquad \mu \in [0, 1].
\end{aligned} \tag{3.4}
$$

Now, from (3.4) and Cauchy's theorem we have first

$$\overline{\lim_{k\to\infty}} \left(\|D^{\gamma}(Qf\tilde{q}_k - Q\tilde{p}_k)\|_{\infty,\, P(0,\, r)}\right)^{1/\Lambda_v(k)} \leqslant \rho_v(r, \mathfrak{D}), \qquad \mu \in [0, 1), \quad \gamma \in \mathbb{N}^d. \tag{3.5}$$

Second, for each $z \in \overline{P(0, R)}$ and $\gamma \in \mathbb{N}^d$ such that if $|\Pi_j(z)| = R_j$ then $\gamma_j = 0$, $1 \leqslant j \leqslant d$; therefore,

$$\overline{\lim_{k\to\infty}} \ |D^{\gamma}(Qf\tilde{q}_k - Q\tilde{p}_k)(z)|^{1/\Lambda_v(k)} \leqslant \rho_v(R, \mathfrak{D}). \tag{3.6}$$

On the other hand, if the assertion (ii) takes place, then from (3.1), (3.2) and since $\Lambda_v(k) \leqslant \sigma_v(k)$ we have

$$\overline{\lim_{k\to\infty}} \left(\|D^{\gamma}(Qf\tilde{q}_k - Q\tilde{p}_k)\|_{\infty,\, P(0,\, r)}\right)^{1/\Lambda_v(k)} \leqslant \rho_v(r, \mathfrak{D}), \qquad \mu \in [0, 1], \quad \gamma \in \mathbb{N}^d. \tag{3.7}$$

Thus, by applying (3.7) for each z_i $(1 \leqslant i \leqslant h)$ with γ taking its values on I_i, we obtain

$$\overline{\lim_{k\to\infty}} \left(|D^{\gamma}(Qf\tilde{q}_k)|\ (z_i)\right)^{1/\Lambda_v(k)} \leqslant \rho_v(\mathscr{Z}_v, \mathfrak{D}) < 1, \qquad \gamma \in \mathrm{I}_i, \quad 1 \leqslant i \leqslant h. \tag{3.8}$$

When the assertion (i) is fulfilled, the same conclusion (3.8) remains valid, after applying (3.6) in the case that $z_i \in \partial P(0, R)$ $(1 \leqslant i \leqslant h)$ and (3.5) to the remaining z_i $(1 \leqslant i \leqslant h)$, again with γ taking its values on I_i in both cases.

Taking into account that

$$D^{\gamma}(Qf\tilde{q}_k)(z_i) = \sum_{\theta \in \mathbb{N}^d,\, \theta \leqslant \gamma} D^{\theta}\tilde{q}_k(z_i) \frac{\gamma!}{(\gamma-\theta)!} D^{\gamma-\theta}(Qf)(z_i), \qquad \gamma \in \mathrm{I}_i, \quad 1 \leqslant i \leqslant h,$$

and if for each i, $1 \leqslant i \leqslant h$, we apply Lemma 3.4, with $\mathrm{I} = \mathrm{I}_i$ and

$$l(\theta, \gamma) = \begin{cases} 0, & \gamma \ngeqslant \theta \\ \dfrac{\gamma!}{(\gamma-\theta)!} D^{\gamma-\theta}(Qf)(z_i), & \gamma \geqslant \theta, \end{cases}$$

we can deduce that there exist constants $l_i^{-1}(\theta, \gamma)$ (with $\theta, \gamma \in \mathrm{I}_i$) such that $D^{\theta}\tilde{q}_k(z_i) = \sum_{\gamma \in \mathrm{I}_i} l_i^{-1}(\theta, \gamma)\, D^{\gamma}(Qf\tilde{q}_k)(z_i)$, $\theta \in \mathrm{I}_i$, $1 \leqslant i \leqslant h$, for every k.

So, from here and (3.8), we have proved that for every $v \in (\mathbb{R}^+)^d$

$$\overline{\lim_{k \to \infty}} (|D^\gamma \tilde{q}_k(z_i)|)^{1/\Lambda_v(k)} \leqslant \rho_v(\mathscr{L}_v, \mathfrak{D}) < 1, \qquad \gamma \in \mathrm{I}_i, \quad 1 \leqslant i \leqslant h. \quad (3.9)$$

Now, let us define for each k, $S_k = \tilde{q}_k - c_k Q$, where $c_k = \tilde{q}_k(0)/Q(0)$ ($= \tilde{q}_{k,\beta_0}/Q_{\beta_0}$, resp.). Thus, $S_k(x) = \sum_{\beta \in M^*} S_{k,\beta} x^\beta$ and since $D^\gamma S_k(z_i) = D^\gamma \tilde{q}_k(z_i)$, $\gamma \in \mathrm{I}_i$, $1 \leqslant i \leqslant h$, one has

$$\sum_{\beta \in M^*} S_{k,\beta} \frac{\beta!}{(\gamma - \gamma)!} (z_i)^{\beta - \gamma} = D^\gamma \tilde{q}_k(z_i), \qquad \gamma \in \mathrm{I}_i, \quad 1 \leqslant i \leqslant h. \quad (3.10)$$

Since we have assumed in (2.1) that the matrix associated to the linear system above, with $(\#M - 1)$ equations and $(\#M - 1)$ unknowns ($S_{k,\beta}$, $\beta \in M^*$), is nonsingular, it implies the existence of some constants $c(\beta, i, \gamma)$, $\gamma \in \mathrm{I}_i$, $1 \leqslant i \leqslant h$, and $\beta \in M^*$ so that

$$S_{k,\beta} = \sum_{i=1}^{h} \sum_{\gamma \in \mathrm{I}_i} c(\beta, i, \gamma)\, D^\gamma \tilde{q}_k(z_i), \qquad \beta \in M^*.$$

Now, (3.9) yields

$$\overline{\lim_{k \to \infty}} (\max_{\beta \in M^*} |S_{k,\beta}|)^{1/\Lambda_v(k)} \leqslant \rho_v(\mathscr{L}_v, \mathfrak{D}).$$

But since

$$\begin{aligned} |1 - c_k \max_{\beta \in M} |Q_\beta|| &= |\max_{\beta \in M} |\tilde{q}_{k,\beta}| - \max_{\beta \in M} |c_k Q_\beta|| \\ &\leqslant \max_{\beta \in M} |\tilde{q}_{k,\beta} - c_k Q_\beta| = \max_{\beta \in M^*} |S_{k,\beta}| \end{aligned}$$

we finally conclude that

$$\overline{\lim_{k \to \infty}} (\|\tilde{q}_k - \tilde{Q}\|)^{1/\Lambda_v(k)} \leqslant \rho_v(\mathscr{L}_v, \mathfrak{D}), \qquad \text{where} \quad \tilde{Q} = Q/\max_{\beta \in M} |Q_\beta|. \quad (3.11)$$

In particular, if only the weak definition works, we obtain (3.11).

On the contrary, if we are in the strong case, (3.11) enables us to obtain

$$\overline{\lim_{k \to \infty}} (\|q_k - Q^*\|)^{1/\Lambda_v(k)} \leqslant \rho_v(\mathscr{L}_v, \mathfrak{D}), \qquad \text{with} \quad Q^* = Q/Q(0). \quad (3.12)$$

Finally, from (3.4) and (3.11) we see that (2.2) holds.

Analogously, in the case corresponding to assertion (ii), (3.7) and (3.11) yield (2.3). ∎

Now, in order to get the proof of the second Montessus-type theorem, we need the following result.

PROPOSITION 3.3. *Let* $(N_k)_{k\in\mathbb{N}}$, $(M_k)_{k\in\mathbb{N}}$, $(E_k)_{k\in\mathbb{N}}$, $(\sigma_u(k))_{k\in\mathbb{N}}$, $(S_u(k))_{k\in\mathbb{N}}$, δ, σ_1 *be as above and* $u\in(\mathbb{R}^+)^d$ *such that* $\lim_{k\to\infty} S_u(k)/\sigma_u(k) = 1$. *Let* $R>0$ *be a polyradius and consider a function* f *in* $\mathbb{C}^d$ *holomorphic in a neighborhood of the origin. Then, if* $(r_k)_{k=1}^{\infty}$ *is a* σ_1*-geometrically strong (weak) OPTA of* f *for* $[(N_k)_{k=1}^{\infty}, (M_k)_{k=1}^{\infty}, R, \delta, \sigma_1]$, *we have that, for any* $\lambda\in\mathbb{R}^+$, $(r_k)_{k=1}^{\infty}$ *is a* σ_1*-geometrically strong (resp. weak) OPTA of* f *for* $[(N_k)_{k=1}^{\infty}, (M_k)_{k=1}^{\infty}, R\lambda^u, \tilde{\delta}, \sigma_1]$, *where* $\tilde{\delta}=(\tilde{\delta}_k)_{k\in\mathbb{N}}\subset[1,\infty)$ *and* $\lim_{k\to\infty}(\tilde{\delta}_k)^{1/\sigma_1(k)}=1$.

Proof. Let $\lambda>0$. For every k, $r_k=p_k/q_k$ and by the definition of OPTA, it holds that

(i) $q_k(x)=\sum_{\beta\in M_k} q_{\beta,k}x^{\beta}$ and $1=q_{0,k}$ (resp. $1=\max_{\beta\in M_k}|q_{\beta,k}|$)

(ii) $\sum_{\alpha\in E_k}|\sum_{\beta\in M_k} q_{\beta,k} f_{\alpha-\beta}R^{\alpha}| \leqslant \delta(k)\min_{c(x)=1}\sum_{\alpha\in E_k}|\sum_{\beta\in M_k} x_\beta f_{\alpha-\beta}R^{\alpha}|$, where $c(x)=x_0$ ($=\max_{\beta\in M_k}|x_\beta|$, resp.)

(iii) p_k is the MacLaurin polynomial of $(q_k f)$ with exponents belonging to N_k.

Let

$$\tilde{\delta}_k=\begin{cases}\delta_k, & \text{if } M_k=\{0\}\\ \delta_k\max\{\lambda,\lambda^{-1}\}^{S_u(k)-\sigma_u(k)}, & \text{if } M_k\neq\{0\}.\end{cases}$$

It is enough to observe that in this case we have

$$\sum_{\alpha\in E_k}\left|\sum_{\beta\in M_k} q_{\beta,k} f_{\alpha-\beta}(R\lambda^u)^{\alpha}\right| \leqslant \tilde{\delta}(k)\min_{c(x)=1}\sum_{\alpha\in E_k}\left|\sum_{\beta\in M_k} x_\beta f_{\alpha-\beta}(R\lambda^u)^{\alpha}\right|$$

with $\tilde{\delta}=(\tilde{\delta}_k)_{k\in\mathbb{N}}\subset[1,\infty)$ and $\lim_{k\to\infty}(\tilde{\delta}_k)^{1/\sigma_1(k)}=1$, and this settles the proof. ∎

Proof (Theorem 2.5). From the previous proposition we get, for any $\lambda\in\mathbb{R}^+$, that $(r_k)_{k=1}^{\infty}$ is a σ_1-geometrically strong (weak) OPTA of f for $[(N_k)_{k=1}^{\infty}, (M_k)_{k=1}^{\infty}, R\lambda^u, \tilde{\delta}, \sigma_1]$. So, if we apply the result of Theorem 2.4, part (ii), then for every $\lambda\in\mathbb{R}^+$ satisfying $\{z_1, ..., z_h\}\subset\overline{P(0, R\lambda^u)}\subset\mathfrak{D}$ and for any $v\in(\mathbb{R}^+)^d$, (2.6) holds, since in this case $\max\{\alpha_i : \alpha\in E_k, 1\leqslant i\leqslant h\}\leqslant \text{const}\cdot\sigma_u(k)(S_u(k)/\sigma_u(k))$ and then the hypothesis in part (ii) is fulfilled.

On the other hand, assume that for any k the polynomials p_k and q_k are normalized in the sense that the rational function $r_k=p_k/q_k=\tilde{p}_k/\tilde{q}_k$ satisfies

$1 = \max_{\beta \in M} |\tilde{q}_{\beta,k}|$, $\tilde{q}_{0,k} \geqslant 0$. Now, proceeding as in proof of Theorem 2.4, we have for any $\lambda \in \mathbb{R}^+$ such that $\overline{P(0, R\lambda^u)} \subset \mathfrak{D}$ and for each $v \in (\mathbb{R}^+)^d$ the following,

$$\overline{\lim_{k \to \infty}} (\| Qf\tilde{q}_k - Q\tilde{p}_k \|_{\infty, P(0,r)})^{1/A_v(k)} \leqslant \rho_v(r, \mathfrak{D}) < 1,$$

where $\mu \in [0, 1]$ and $r = R\lambda^u \mu^v$.

Therefore, from this and (2.6), we easily conclude (2.5). ∎

4. EXAMPLES AND DISCUSSION

4.1. *Numerical Example*

In order to check the numerical performance of these convergence results, we consider the following simple example, taken from [14]. Let $f(x, y) = \exp(x+y)/(1-2(x+y)+x^2+y^2)$, which is holomorphic in $P(0, (1-\frac{1}{2}\sqrt{2}, 1-\frac{1}{2}\sqrt{2}))$, with fQ being holomorphic in $\mathbb{C}^2$, when we take $Q(x, y) = 1-2(x+y)+x^2+y^2$. For each $n \in \mathbb{N}$ consider the sets $N_n = M_n = \{\alpha \in \mathbb{N}^2 : \alpha_1 + \alpha_2 \leqslant n\}$ and for $n, m \in \mathbb{N}$ and $s \in (0, \infty)$, denote by $r_{n,m;s}$ the unique (in this case) rational function for which the conditions in Definition 2.2 hold, with $N = N_n$, $M = M_m$, $R = (s, s)$, so that in this case the denominator vector in the requirement (b) in Definition 2.2 is taken as a strong pseudominimum of T for $[\#M, \#E, 1]$ with respect to the norm $\|\cdot\|_2$ in $\mathbb{C}^{\#E}$. From Remark 2.9, $(r_{n,m;s})_{n \in \mathbb{N}}$ is a σ_1-geometrically strong OPTA of f for $[(N_n)_{n \in \mathbb{N}}, (M_m)_{n \in \mathbb{N}}, R, \delta', \sigma_1]$, with σ_1 as in Theorem 2.4 and δ' as in Proposition 2.1. It is easy to check that for $m = 2$ the hypothesis in Theorem 2.5 with $u = (1, 1)$ is fulfilled. These choices for the sets N, M, and R are the most natural if we take into account the symmetry properties of f.

Under these conditions, the following results are obtained for the error of the sequences of OPTA, for $m = 2$ (Table I) and $m = 3$ (Table II). The results are displayed for different values of the radius s (1, 4, 12).

As we can see from Table I, the exhibited results are somewhat better than the corresponding for the classical Multivariate PA, which are exhibited in [14, p. 55, Tables 1 and 3]. However, let us note that there exist some differences between the notation used in our tables and the corresponding notation in [14].

Thus, in order to make comparisons between the respective tables, take into account that the values of n in our tables correspond to $(\frac{(n+1)(n+2)}{2} - 1)$ in [14]. On the other hand, the goodness of these results seems to be rather independent of the choice of the polyradius. As for Table II, we remark that even for a quite unsuitable choice of the denominator lattices, the results are rather good. The convergence of

TABLE I

n	$(f-r_{n,2;1})(1,1)$	$(f-r_{n,2;4})(1,1)$	$(f-r_{n,2;12})(1,1)$
0	−0.753191324E+01	−0.753191324E+01	−0.753191324E+01
1	−0.874836701E+01	−0.875497514E+01	−0.875544284E+01
2	−0.552540039E+02	0.881535990E+00	−0.224459204E+01
3	0.957663093E+00	−0.818541129E+00	−0.107603082E+01
4	−0.307568600E−00	−0.394515264E+00	−0.401465328E+00
5	−0.125465270E−00	−0.125636443E+00	−0.125677852E+00
6	−0.349880409E−01	−0.343300978E−01	−0.342883745E−01
7	−0.846778958E−02	−0.828820676E−02	−0.827667656E−02
8	−0.183211566E−02	−0.179161859E−02	−0.178901494E−02
9	−0.358525976E−03	−0.350374245E−03	−0.349850073E−03
10	−0.640205685E−04	−0.625317488E−04	−0.624360915E−04
11	−0.105091752E−04	−0.102602184E−04	−0.102440798E−04
12	−0.159656657E−05	−0.155913769E−05	−0.155508795E−05
13	−0.226181362E−06	−0.220943222E−06	−0.220474174E−06
14	−0.221577183E−07	−0.241084921E−07	−0.236085231E−07
15	0.303044327E−07	0.321656835E−07	0.285727975E−08
16	0.545424923E−07	−0.687935140E−07	−0.375821454E−07
17	−0.234716173E−06	−0.162879244E−06	0.446093242E−07
18	−0.123789808E−05	−0.692882002E−06	−0.683911286E−06
19	−0.128684283E−06	−0.379612555E−06	−0.167479394E−05

TABLE II

n	$(f-r_{n,3;1})(1,1)$	$(f-r_{n,3;4})(1,1)$	$(f-r_{n,3;12})(1,1)$
0	−0.723116136E+01	−0.723116136E+01	−0.723116136E+01
1	−0.584786727E+01	−0.559539083E+01	−0.400313471E+01
2	−0.241274170E+01	−0.213682670E+01	−0.108565767E+01
3	−0.430026584E+00	−0.108234267E+00	−0.119129119E+00
4	−0.310332721E−01	−0.185514007E−01	−0.425884872E−01
5	0.716607389E−02	−0.385697561E−02	−0.122355182E−01
6	0.179205232E−02	−0.837947739E−03	−0.300630263E−02
7	0.327725049E−03	−0.159032193E−03	−0.649699640E−03
8	0.546155209E−04	−0.267938175E−04	−0.125646903E−03
9	0.847719373E−05	−0.406991798E−05	−0.220106761E−04
10	0.123011316E−05	−0.562937033E−06	−0.352249011E−05
11	0.165575917E−06	−0.784824596E−07	−0.545905424E−06
12	0.221356755E−07	0.583972515E−08	−0.357991858E−08
13	0.811308887E−08	0.416757366E−07	0.478952993E−07
14	0.229872295E−06	−0.964958558E−07	−0.138468278E−06
15	−0.256567629E−07	−0.245392089E−07	0.273429315E−07
16	0.539337464E−10	−0.667729427E−08	0.811262408E−07
17	−0.564969564E−07	−0.482506479E−09	0.603265882E−09
18	0.512957809E−06	−0.335587305E−06	−0.332612462E−06
19	−0.385226670E−06	−0.346397386E−06	0.119421822E−05

TABLE III

n	$q_{n,2;4}(x, y)$
0	$1 - 3.00000000000000(x+y) + 0.500000000000000\text{E}+01xy + 3.50000000000000(x^2+y^2)$
1	$1 - 2.45790872328134(x+y) + 0.187168110918544\text{E}+01xy + 1.78499133448874(x^2+y^2)$
2	$1 - 2.09920848574370(x+y) + 0.383637716865703\text{E}+00xy + 1.14799421194779(x^2+y^2)$
3	$1 - 2.01507848661464(x+y) + 0.586774253105235\text{E}-01xy + 1.02114554139931(x^2+y^2)$
4	$1 - 2.00046756928326(x+y) + 0.175172362022106\text{E}-02xy + 1.00056999703746(x^2+y^2)$
5	$1 - 2.00000211380682(x+y) + 0.461682456266589\text{E}-06xy + 0.999993775583263(x^2+y^2)$
6	$1 - 1.99999846390479(x+y) - 0.664319372676843\text{E}-05xy + 0.999997094262517(x^2+y^2)$
7	$1 - 1.99999982389490(x+y) - 0.732247799315206\text{E}-06xy + 0.999999700263661(x^2+y^2)$
8	$1 - 1.99999998570868(x+y) - 0.585568543661120\text{E}-07xy + 0.999999976668479(x^2+y^2)$
9	$1 - 1.99999999901649(x+y) - 0.399759421680475\text{E}-08xy + 0.999999998435988(x^2+y^2)$
10	$1 - 1.99999999994013(x+y) - 0.242167277692631\text{E}-09xy + 0.999999999906603(x^2+y^2)$
11	$1 - 1.99999999999673(x+y) - 0.131828326370219\text{E}-10xy + 0.999999999994972(x^2+y^2)$
12	$1 - 1.99999999999981(x+y) - 0.750277942702316\text{E}-12xy + 0.999999999999720(x^2+y^2)$
13	$1 - 1.99999999999998(x+y) - 0.936173068631420\text{E}-13xy + 0.999999999999969(x^2+y^2)$
14	$1 - 1.99999999999993(x+y) - 0.293629818896257\text{E}-12xy + 0.999999999999901(x^2+y^2)$
15	$1 - 2.00000000000009(x+y) + 0.376188309899084\text{E}-12xy + 1.00000000000012(x^2+y^2)$
16	$1 - 1.99999999999995(x+y) - 0.228039274587000\text{E}-12xy + 0.999999999999927(x^2+y^2)$
17	$1 - 2.00000000000009(x+y) + 0.367614445174061\text{E}-12xy + 1.00000000000012(x^2+y^2)$
18	$1 - 1.99999999999992(x+y) - 0.322318695351547\text{E}-12xy + 0.999999999999893(x^2+y^2)$
19	$1 - 1.99999999999990(x+y) - 0.423096355757701\text{E}-12xy + 0.999999999999860(x^2+y^2)$

sequences of OPTA with denominator lattice larger than the set of exponents of Q will be treated in a forthcoming paper.

On the other hand, Table III shows the convergence to Q of the denominators of one of the sequences of OPTA used in Table I. This convergence appears to be faster than the corresponding convergence for the denominators of multivariate PA [14, p. 56, Table 2]. Finally, let us remark that the numerical experiments above were carried out with Microsoft Fortran Power Station.

4.2. *Counterexamples*

To end this paper we shall show the sharpness, in a certain sense, of our main results by means of two rather simple examples.

EXAMPLE 4.1. This first counterexample is related to Theorem 2.4, part (i). Using the same notations as in the numerical example above, let $(p_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{N}\backslash\{0\}$ so that $\lim_{k\to\infty} p_k^{1/k} = \infty$ and consider for each k and $j \geqslant 3$ the sequence of lattices

$$N_k = \{\alpha \in \mathbb{N}^2 : |\alpha| \leqslant k\} \cup \{\alpha \in \mathbb{N}^2 : \alpha \leqslant (k, j+1)\}$$
$$\cup \{\alpha \in \mathbb{N}^2 : \alpha \leqslant (1, kp_k) \text{ or } \alpha \leqslant (k^2 p_k, 1)\}.$$

Let $M=\{(0,0),(1,0),(0,1),(1,1)\}$, $R=(R_1,R_2)$ and $v=(v_1,v_2)\in(\mathbb{R}^+)^2$, with $v_1\leqslant v_2$. In this case $\lim_{k\to\infty}\Lambda_v(k)/k=\lim_{k\to\infty}\sigma_v(k)/k=v_1$.

Now, if we consider the function $f(z,w)=(1/(1-az))+(1/(1-bw))+(w^j/(1-cz))$, where

$$\begin{cases} a=R_1^{-1}, \quad b=R_2^{-1}, \quad c\in(0,a) \\ (z,w)\in(\mathbb{C}\backslash\{a^{-1},c^{-1}\})\times(\mathbb{C}\backslash\{b^{-1}\}) \end{cases}$$

and take the polynomial $Q(z,w)=(1-az)(1-bw)$, we have immediately that $\mathfrak{D}=\{|z|<c^{-1}\}\times\mathbb{C}$, $\overline{P(0,R)}\subset\mathfrak{D}$ and $\rho_v(R,\mathfrak{D})=(ca^{-1})^{1/v_1}<1$.

Thus, in order to construct a sequence $(r_k)_{k=1}^{\infty}$ being σ_1-geometrically strong OPTA of f for $[(N_k)_{k=1}^{\infty},(M)_{k=1}^{\infty},\delta,R,\sigma_1]$ (with δ_k, to be determined, taken as in Theorem 2.4), we must take a sequence of polynomials $(q_k)_{k=1}^{\infty}$ with exponent sets given by M, verifying that $q_k(0)=1$ and so that if we denote $q_k(z,w)=\sum_{\beta\in M}q_{k,\beta}(z,w)^{\beta}$, then

$$\|T_k((q_{k,\beta})_{\beta\in M})\|_1\leqslant\delta_k\min_{x_0=1}\|T_k((x_\beta)_{\beta\in M})\|_1 \tag{4.1}$$

where

$$\begin{cases} f(z,w)=\sum_{\alpha\in\mathbb{N}^2}f_\alpha\cdot(z,w)^\alpha \\ T_k((x_\beta)_{\beta\in M})=\left(\sum_{\beta\in M}x_\beta f_{\alpha-\beta}\right)_{\alpha\in E_k}. \end{cases}$$

But if we take the sets $E_k^*=\{\alpha\in E_k:\text{there exists }\beta\in M\text{ for which }f_{\alpha-\beta}\neq0\}$, it is clear that solving the problem (4.1) above is equivalent to solving the following

$$\|T_k^*((q_{k,\beta})_{\beta\in M})\|_1\leqslant\delta_k\min_{x_0=1}\|T_k^*((x_\beta)_{\beta\in M})\|_1 \tag{4.2}$$

with $T_k^*((x_\beta)_{\beta\in M})=(\sum_{\beta\in M}x_\beta f_{\alpha-\beta})_{\alpha\in E_k^*}$.

Now, it is easy to see that $\#E_k^*=6$ for k sufficiently large. Therefore, taking $(q_k)_{k=1}^{\infty}$ so that

$$\|T_k^*((q_{k,\beta})_{\beta\in M})\|_1=\min_{x_0=1}\|T_k^*((x_\beta)_{\beta\in M})\|_2 \qquad\text{and}\qquad q_{k,0}=1 \tag{4.3}$$

we have, from Remark 2.5, that (4.2) holds, and so (4.1) does, with $\delta_k=6$. Problem (4.3) is equivalent to solving the system

$$\begin{cases} \sum_{\beta\in M}q_{\beta,k}\sum_{\alpha\in E_k^*}f_{\alpha-\beta}\overline{f_{\alpha-\theta}}R^{2\alpha}=0, \qquad \gamma\in M\backslash\{0\} \\ q_{k,0}=1 \end{cases}$$

whose unique solution, if we denote $e = ca^{-1}$, is given by

$$q_{(1,0),k} = \frac{-a(1+\frac{1}{2}b^{-2j}(1+e+e^2)\,e^{2k}+\frac{1}{2}b^{-4j}(1-e+e^2)\,e^{4k+1})}{1+(\frac{7}{4}-e+\frac{3}{4}e^2)\,b^{-2j}e^{2k}+\frac{1}{2}b^{-4j}(1-e+e^2)\,e^{4k}}$$

$$q_{(0,2),k} = \frac{-b(1+(1+\frac{e}{2})\,b^{-2j}e^{2k}+\frac{1}{4}b^{-4j}(1+e^2)\,e^{4k})}{1+(\frac{7}{4}-e+\frac{3}{4}e^2)\,b^{-2j}e^{2k}+\frac{1}{2}b^{-4j}(1-e+e^2)\,e^{4k}}$$

$$q_{(1,1),k} = \frac{ab(1+(\frac{1}{4}+e+\frac{1}{4}e^2)\,b^{-2j}e^{2k}+\frac{1}{4}b^{-4j}(1+e^2)\,e^{4k+1})}{1+(\frac{7}{4}-e+\frac{3}{4}e^2)\,b^{-2j}e^{2k}+\frac{1}{2}b^{-4j}(1-e+e^2)\,e^{4k}}.$$

from this, it is clear that

$$\lim_{k\to\infty} \|Q-q_k\|^{1/k} = (ca^{-1})^2 < 1. \tag{4.4}$$

So, if we consider

$$\begin{aligned} p_k(z,w) &= 2 + q_{(0,1),k}w + q_{(0,1),k}z + \left(1+\frac{q_{(0,1),k}}{b} + z\left(q_{(1,0),k}+\frac{q_{(1,1),k}}{b}\right)\right) \\ &\quad\times \sum_{n=1}^{kp_k} (bw)^n + \left(1+\frac{q_{(1,0),k}}{a} + w\left(q_{(0,1),k}+\frac{q_{(1,1),k}}{a}\right)\right) \\ &\quad\times \sum_{n=1}^{k^2p_k} (az)^n + w^3(1+q_{(0,1),k}w) \\ &\quad + w^3\left(1+\frac{q_{(1,0),k}}{c} + w\left(q_{(0,1),k}+\frac{q_{(1,1),k}}{c}\right)\right)\sum_{n=1}^{k} (cz)^n \end{aligned} \tag{4.5}$$

It is easily seen that, for any k, p_k is the Taylor polynomial of fQ corresponding to the finite set N_k and it implies that the sequence of rational functions $(r_k)_{k=1}^{\infty}$, with $r_k = p_k/q_k$, is a sequence of σ_1-geometrically strong OPTA of f for $[(N_k)_{k=1}^{\infty}, (M)_{k=1}^{\infty}, \delta, R, \sigma_1]$. Thus, since the hypotheses in Theorem 2.4, part (i), are fulfilled, if for each $\mu \in [0,1]$ we denote $r = R\mu^v$, then we have for any $\varepsilon > 0$

$$\overline{\lim_{k\to\infty}} (\|f-r_k\|_{\infty, \overline{P(0,r)}\setminus L_\varepsilon})^{1/\Lambda_v(k)} \leqslant \rho_v(r, \mathfrak{D}) = \mu(ca^{-1})^{1/v_1} \tag{4.6}$$

and if $\mu \neq 1$, for every $\gamma \in \mathbb{N}^2$

$$\overline{\lim_{k\to\infty}} (\|D^\gamma(f-r_k)\|_{\infty, \overline{P(0,r)}\setminus L_\varepsilon})^{1/\Lambda_v(k)} \leqslant \rho_v(r, \mathfrak{D}). \tag{4.7}$$

(4.8) Therefore, for $(z, w) \in \partial P(0, R)$ and $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$ taken so that if $|z| = R_1$ ($|w| = R_2$) we have $\gamma_1 = 0$ (resp. $\gamma_2 = 0$), then

$$\overline{\lim_{k \to \infty}} |D^\gamma (f - r_k)(z, w)|^{1/\Lambda_v(k)} \leqslant \rho_v(R, \mathfrak{D}). \tag{4.9}$$

For $(z, w) \in \overline{P(0, R)} \backslash Q^{-1}\{0\}$ it is easily seen that

$$(fq_k - p_k)(z, w) = \frac{(az^{k^2 p_k + 1})}{1 - az} \left(1 + \frac{q_{(1,0),k}}{a} + w\left(q_{(0,1),k} + \frac{q_{(1,1),k}}{a}\right)\right)$$
$$+ \frac{(bw)^{kp_k + 1}}{1 - bw} \left(1 + \frac{q_{(0,1),k}}{b} + z\left(q_{(1,0),k} + \frac{q_{(1,1),k}}{b}\right)\right)$$
$$+ \left(1 + \frac{q_{(1,0),k}}{c} + w\left(q_{(0,1),k} + \frac{q_{(1,1),k}}{c}\right)\right) + \frac{(cz)^{k+1}}{1 - cz} w^j.$$

From this and (4.5) we have the following:

- The equality is attained in (4.6). Moreover, if $\gamma_2 \leqslant j$ (resp. $\gamma_2 = j$ or $0 \leqslant \gamma_2 < j$ and $w \neq 0$) in (4.7) (resp. (4.9)) the equality also holds.
- For $(z, w) \in \mathbb{C}^2 \backslash (\overline{P(0, R)} \cup Q^{-1}\{0\})$ we have that $\lim_{k \to \infty} r_k(z, w) = \infty$.
- Finally, for $(z, w) \in \mathbb{C}^2 \backslash (P(0, R) \cup Q^{-1}\{0\})$ and $\gamma \in \mathbb{N}^2$ not satisfying (4.8), we have that $\lim_{k \to \infty} D^\gamma r_k(z, w) = \infty$.

Therefore, from the considerations above we conclude that the region of convergence of the sequence of OPTA $(r_k)_{k=1}^\infty$ is almost as small as the corresponding region of convergence for the Taylor series for f, but the rate of this convergence is similar to that obtained for the Taylor series of fQ.

EXAMPLE 4.2. This final counterexample is displayed in order to show the sharpness of the conditions of Theorem 2.4, part (ii).

Indeed, under the same notations as above, let $(p_k)_{k=1}^\infty$ be a sequence of positive integers with $\lim_{k \to \infty} p_k^{1/k} = p_0 > 1$, $M = \{(0, 0), (1, 0), (2, 0)\}$, $R = (R_1, R_2) \in (\mathbb{R}^+)^2$, $v = (v_1, v_2) \in (\mathbb{R}^+)^2$, and for $k \in \mathbb{N} \backslash \{0\}$ consider the sequence of lattices

$$N_k = \{\alpha \in \mathbb{N}^2 : |\alpha| \leqslant k\} \backslash \{(0, k)\} \cup \{\alpha \in \mathbb{N}^2 : \alpha \leqslant (kp_k, 0)\}.$$

From this, it is clear that

$$\overline{\lim_{k \to \infty}} (\max\{\alpha_i : \alpha \in E_k, i = 1, 2\})^{1/\sigma_1(k)} = p_0^{1/\min\{v_1, v_2\}} > 1. \tag{4.5}$$

Now, taking the function $f(z, w) = az/((1-az)^2) + 1/(1-bw)$, where

$$\begin{cases} a = R_1^{-1} \\ b \in (0, R_2^{-1}) : bR_2 p_0 > 1 \\ (z, w) \in (\mathbb{C} \backslash \{a^{-1}\}) \times (\mathbb{C} \backslash \{b^{-1}\}) \end{cases}$$

and the polynomial $Q(z, w) = (1-az)^2$, we can see, similarly as in Example 4.1, that $\# E_k^* = 3$. Thus, if we choose the sequence $(q_k)_{k=1}^{\infty}$ such that

$$\|T_k^*((q_{k,\beta})_{\beta \in M})\|_2 = \min_{x_0 = 1} \|T_k^*((x_\beta)_{\beta \in M})\|_2 \qquad \text{and} \qquad q_{k,0} = 1$$

and for each k, take p_k as the Taylor polynomial of fQ corresponding to the lattice N_k, then it is easy to see that $(r_k)_{k=1}^{\infty}$, where $r_k = p_k/q_k$, is a sequence of σ_1-geometrically strong OPTA of f for $[(N_k)_{k=1}^{\infty}, (M)_{k=1}^{\infty}, \delta, R, \sigma_1]$.

On the other hand, it is immediate to check that the requirements in Theorem 2.4 are fulfilled, except the corresponding requirements to part (ii). The sharpness of this condition is pointed out by taking into account that in this case the coefficients of the denominators q_k are given by

$$q_{(2,0),k} = \frac{a^4}{(kp_k(bR_2)^{k-1})^2 + a^2/2}$$

$$q_{(1,0),k} = -2a + \frac{2a}{(kp_k(bR_2)^{k-1})^{-2} a^2 + 2}$$

and hence $\lim_{k \to \infty} q_k(z) = 1 - az \neq Q$, showing that the sequence of denominators of OPTA does not converge to the polynomial Q.

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The author expresses his gratitude to the anonymous referees, whose suggestions have improved the quality of the article. In particular, one of the referees pointed out the existence of a very recent paper (see [22]), which appeared when this paper was submitted and where the least squares approach is used to construct multivariate rational approximants. However, the theory developed in the present paper is more general and contains the approach employed in [22] as a particular case, since, in practice, when a σ_1-geometrically strong OPTA of f is computed, using Remark 2.9, we can replace the norm $\| \ \|_1$ by $\| \ \|_2$ (taking $\delta = 1$), as we carried out in the numerical examples displayed in Section 4.

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